

HOOK MODULES FOR GENERAL LINEAR GROUPS

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ABSTRACT. For an arbitrary infinite field k of characteristic $p > 0$, we describe the structure of a block of the algebraic monoid $M_n(k)$ (all $n \times n$ matrices over k), or, equivalently, a block of the Schur algebra $S(n, p)$, whose simple modules are indexed by p -hook partitions. The result is known; we give an elementary and self-contained proof, based only on a result of Peel and Donkin's description of the blocks of Schur algebras. The result leads to a character formula for certain simple $GL_n(k)$ -modules, valid for all n and all p . This character formula is a special case of one found by Brundan, Kleshchev, and Suprunenko and, independently, by Mathieu and Papadopoulos.

INTRODUCTION

We describe the structure of the family of Weyl modules labeled by p -hooks, for $GL_n(k)$ where k is an infinite field of characteristic $p > 0$. The main result, given in the Theorem of Section 2, leads to a description of the module structure of the family of projective-injective tilting modules labeled by p -hooks, and determines completely the corresponding block of $M_n(k)$. The result was previously obtained in [8, §5.7] by a different argument; see also [12]. Another consequence of the Theorem is a character formula for simple $GL_n(k)$ -modules labeled by p -hooks; this character formula was previously conjectured by Jantzen. After the first draft of this paper was written, it was pointed out to us that the character formula is an easy consequence of a result of Brundan, Kleshchev, and Suprunenko [2, Theorem 6.3] describing the character of simple modules labeled by ‘completely splittable’ weights. The same character formula was obtained independently by Mathieu and Papadopoulos [18] by a different method, and it can be used to give yet another proof of the Theorem of this note.

The idea behind the proof given here is to compare with the symmetric group Σ_p on p letters using the idempotent ‘Schur functor’ of

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[11, §6], given by $M \rightarrow eM$ where e is projection onto the (1^p) weight space. The key lemma (Lemma 2) states that the p -hooks index the simple modules in a block of the Schur algebra $S(n, p)$, for $n \geq p$. This provides a tight upper bound on the number of composition factors of a Weyl module labeled by a given p -hook; the corresponding lower bound is provided by Jantzen's sum formula (see Lemma 1).

We note that the main result also provides a set of examples where

$$\mathrm{Ext}_{\mathrm{GL}_n(k)}^1(L(\lambda), L(\mu)) \simeq \mathrm{Ext}_{\Sigma_p}^1(eL(\lambda), eL(\mu)).$$

The question of determining pairs λ, μ for which such an equality holds has been studied in [9], [16], [15].

1. NOTATION

We mostly follow the notational conventions of [11, §1]. Fix an infinite field k of positive characteristic p , and set $G = \mathrm{GL}_n(k)$, the general linear group of invertible $n \times n$ matrices over k . Let E_{ij} ($1 \leq i, j \leq n$) denote the matrix units in $\mathbf{M}_n(k)$ (the algebraic monoid of $n \times n$ matrices over k). So E_{ij} is the $n \times n$ matrix with (i, j) -entry 1 and all other entries 0. The E_{ij} form a k -basis of $\mathbf{M}_n(k)$; let c_{ij} ($1 \leq i, j \leq n$) be the dual basis of $\mathbf{M}_n(k)^*$. The coordinate algebra $k[G]$ is generated by the c_{ij} and $\det(c_{ij})^{-1}$.

Let T be the subgroup of G consisting of all diagonal matrices. The restrictions $\varepsilon_i = c_{ii}|_T$ ($1 \leq i \leq n$) form a basis of the character group $X = X(T) = \mathrm{Hom}(T, G_m)$ (algebraic group homomorphisms to the multiplicative group G_m); we identify elements of X with elements of \mathbb{Z}^n via the correspondence

$$\lambda = \sum_i \lambda_i \varepsilon_i \leftrightarrow (\lambda_1, \dots, \lambda_n).$$

The root system associated to the pair (G, T) is the set $R = \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\}$. We choose as positive roots the system $R^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}$; then $S = \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq n-1\}$ is the corresponding set of simple roots. Let B be the Borel subgroup of G corresponding with the negative roots. The set of dominant weights is the set

$$X^+ = \{\lambda \in X : \langle \lambda, \alpha^\vee \rangle \geq 0, \text{ all } \alpha \in S\};$$

in terms of our identification $X \simeq \mathbb{Z}^n$ we have that $\lambda = (\lambda_1, \dots, \lambda_n) \in X^+$ if and only if

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

The Weyl group W associated to (G, T) is naturally isomorphic to the symmetric group Σ_n on n letters, via the correspondence $s_\alpha \rightarrow (i, j)$ when $\alpha = \varepsilon_i - \varepsilon_j$. As usual, W acts on X via the rule $s_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$;

we shall need the ‘dot action’ given by $s_\alpha \cdot \lambda = s_\alpha(\lambda + \rho) - \rho$ where $\rho = \sum_i (n - i)\varepsilon_i$. (Here ρ is not the usual ‘half sum of the positive roots,’ which is not always defined in X , but it has the crucial property $\langle \rho, \beta^\vee \rangle = 1$ for all $\beta \in S$.) We also need to consider elements of the affine Weyl group W_p generated by all $s_{\alpha, ap}$ ($\alpha \in R$, $a \in \mathbb{Z}$) where $s_{\alpha, ap} = s_\alpha + ap\alpha$.

Let k_λ (for $\lambda \in X$) be the one dimensional B -module such that T acts via the character λ and the unipotent radical of B acts trivially. For $\lambda \in X^+$ we have the following rational G -modules:

$$\nabla(\lambda) = \text{ind}_B^G k_\lambda$$

$$\Delta(\lambda) = {}^\tau \nabla(\lambda) \simeq \nabla(-w_0\lambda)^* \text{ (the Weyl module)}$$

$$T(\lambda) = \text{indecomposable tilting module of highest weight } \lambda.$$

Here $T(\lambda)$ is the unique indecomposable module of highest weight λ which admits both a Δ and ∇ filtration (see [5]), τ is the ‘transpose dual’ operator ([14, Part II, 2.13]) and w_0 is the longest element of W .

Any finite-dimensional G -module M is the direct sum of its weight spaces: $M = \bigoplus_{\lambda \in X} M_\lambda$ where $M_\lambda = \{m \in M : tm = \lambda(t)m, \text{ all } t \in T\}$. We set

$$\text{ch } M = \sum_\lambda (\dim M_\lambda) e(\lambda) \in \mathbb{Z}[X]$$

(the formal character of M). Here $\mathbb{Z}[X]$, the group ring of X , is the free \mathbb{Z} -module with basis $\{e(\lambda) : \lambda \in X\}$ and multiplication $e(\lambda)e(\mu) = e(\lambda + \mu)$.

A $\text{GL}_n(k)$ -module M is termed *polynomial* if it lifts to a rational $\text{M}_n(k)$ -module. It is well known (see e.g. [11, §1]) that restriction from $\text{M}_n(k)$ to $\text{GL}_n(k)$ induces an full embedding of the category $\text{M}_n(k)\text{-mod}$ of rational left $\text{M}_n(k)$ -modules in the category $\text{GL}_n(k)\text{-mod}$ of rational left $\text{GL}_n(k)$ -modules. The weights of an object $M \in \text{M}_n(k)\text{-mod}$ are of the form $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$. Such weights are often called polynomial weights. In the language of combinatorics, polynomial weights are compositions (i.e. unordered partitions) of length n and dominant polynomial weights are partitions of length at most n . We omit any zero parts when writing a partition, as usual.

The category of polynomial $\text{GL}_n(k)$ -modules is graded by homogeneous degree; that is we have an equivalence

$$\text{M}_n(k)\text{-mod} \simeq \bigoplus_{r \geq 0} S(n, r)\text{-mod}$$

in the sense that every rational $\text{M}_n(k)$ -module may be written as a direct sum of homogeneous ones of various degrees. Here $S(n, r)$ is the Schur algebra in degree r ; its module category $S(n, r)\text{-mod}$ is equivalent with the full subcategory of $\text{M}_n(k)\text{-mod}$ consisting of homogeneous

modules of degree r . We write $\lambda \vdash r$ to indicate that λ is a partition of r (i.e., $\sum \lambda_i = r$); the set of dominant weights for $S(n, r)$ -mod is precisely the set of all $\lambda \vdash r$ such that $\text{length} \leq n$.

Assume that $n \geq r$ for the remainder of this section. Then the set of $\lambda \vdash r$ with $\text{length} \leq n$ is the same as the set of partitions of r , and we have a so-called ‘Schur functor’ from $S(n, r)$ -mod to $k\Sigma_r$ -mod, where Σ_r is the symmetric group on r letters. This is an exact covariant functor given by the rule $M \mapsto eM$ where $e \in S(n, r)$ is the idempotent projector onto the (1^r) -weight space. See [11, Chapter 6] or [17, Chapter 4] for details. In particular we have

$$(1) \quad e\Delta(\lambda) \simeq S^\lambda, \quad e\nabla(\lambda) \simeq S_\lambda$$

for any partition $\lambda \vdash r$, where S^λ (resp., S_λ) is the Specht (resp., dual Specht) module indexed by λ . Here S^λ may be defined (following [13, §4]) as the submodule of M^λ spanned by polytabloids of type λ , where M^λ is the transitive permutation module $\text{ind}_{\Sigma_\lambda}^{\Sigma_r} k$ associated with the Young subgroup Σ_λ corresponding with λ , and $S_\lambda \simeq (S^\lambda)^*$ as $k\Sigma_r$ -modules.

A partition $\lambda \vdash r$ is p -regular if there does not exist an i such that $\lambda_{i+1} = \dots = \lambda_{i+p}$ and column p -regular if its conjugate λ' is p -regular. Equivalently, λ is column p -regular if and only if $\lambda_i - \lambda_{i+1} < p$ for all i . If λ is p -regular then S^λ has a unique top composition factor D^λ ; similarly if λ is column p -regular then S_λ has a unique top composition factor D_λ . The set

$$\{D^\lambda : \lambda \vdash r, \lambda \text{ } p\text{-regular}\} \simeq \{D_\lambda : \lambda \vdash r, \lambda \text{ column } p\text{-regular}\}$$

gives a complete set of isomorphism classes of simple $k\Sigma_r$ -modules. Since we have an isomorphism $S^{\lambda'} \simeq S_\lambda \otimes \text{sgn}$ it follows that the two labellings $\{D^\lambda\}, \{D_\lambda\}$ are related by

$$(2) \quad D^{\lambda'} \simeq D_\lambda \otimes \text{sgn}$$

for any column p regular $\lambda \vdash r$. We also have

$$(3) \quad eL(\lambda) \simeq D_\lambda$$

for all column p -regular partitions $\lambda \vdash r$. Finally, since tensoring by the one dimensional sign representation must take simples to simples, we have $D^\lambda \otimes \text{sgn} \simeq D^{\text{Mull}(\lambda)}$ for all p -regular $\lambda \vdash r$, where $\text{Mull}(\lambda)$ is given by a combinatorial procedure described in [19] and proved in [10].

2. RESULTS

Let $\lambda^i = (p - i, 1^i) = (p - i)\varepsilon_1 + \sum_{j=2}^{i+1} \varepsilon_j$ for $0 \leq i \leq p - 1$. These are the p -hook partitions.

Lemma 1. *For $0 \leq i < \min(n - 1, p - 1)$ the Weyl module $\Delta(\lambda^i)$ has at least two composition factors, of highest weight λ^i and λ^{i+1} .*

Proof. (We are grateful to Jens Jantzen for sending us this argument). It is enough to show that $L(\lambda^{i+1})$ is a composition factor of $\Delta(\lambda^i)$ for $i < \min(n - 1, p - 1)$. Since $\lambda^i - \lambda^{i+1}$ is equal to the sum of the first i simple roots, one can reduce to the root system of type A_i . In that case Jantzen's sum formula [14, Part II, 8.19] has just one term, namely the Weyl character of λ^{i+1} . The result follows. \square

Let r be an arbitrary natural number. Given a partition $\lambda \vdash r$ of length $\leq n$ let us denote by $d(\lambda)$ the maximum of all $d \geq 0$ such that $\lambda_i - \lambda_{i+1} \equiv -1 \pmod{p^d}$ for all $1 \leq i < n$. We need S. Donkin's result from [6], which states that for partitions $\lambda, \mu \vdash r$ of length $\leq n$, the corresponding simple modules $L(\lambda), L(\mu)$ lie in the same block for $S(n, r)$ if and only if both conditions (B1) and (B2) below hold:

- (B1) $d(\lambda) = d(\mu)$ (say $d = d(\lambda) = d(\mu)$);
- (B2) there exists $w \in W$ such that $\lambda_i - i \equiv \mu_{w(i)} - w(i) \pmod{p^{d+1}}$ for all $1 \leq i \leq n$.

Noting that λ and μ will satisfy (B2) if and only if $\lambda + (n^n)$ and $\mu + (n^n)$ also satisfy (B2), we see that (B2) is equivalent to the condition

- (B2') there exists $w \in W$ such that $\lambda + \rho \equiv w(\mu + \rho) \pmod{p^{d+1}}$.

Here, for n -part compositions $\lambda' = (\lambda'_1, \dots, \lambda'_n)$, $\lambda'' = (\lambda''_1, \dots, \lambda''_n)$ we declare that $\lambda' \equiv \lambda'' \pmod{N}$ if and only if $\lambda'_i \equiv \lambda''_i \pmod{N}$ for all $1 \leq i \leq n$.

Lemma 2. *Assume that $n \geq p$. We identify a block with the set of highest weights labeling its simple modules. With that identification, the block of the Schur algebra $S(n, p)$ containing the one row partition $\lambda^0 = (p)$ consists of all the p -hook partitions, and only those partitions.*

Proof. We may assume without loss of generality that $n = p$ since the block is the same for larger n . We have $d = d(\lambda^i) = 0$ for all $0 \leq i \leq p - 1$. Set $\lambda = \lambda^0$. The modulo p residues of the parts of $\lambda + \rho$ in order are $p - 1, p - 2, \dots, 1, 0$. Thus, in order that a partition $\mu \vdash p$ satisfy condition (B2') in relation to λ , it is necessary and sufficient that the modulo p residues of $\mu + \rho$ are pairwise distinct.

It is easy to check that this condition holds true for all the p -hook partitions λ^i . In fact, one checks by direct calculation that $\lambda^i + \rho \equiv s_\alpha(\lambda^{i+1} + \rho)$ for all $0 \leq i < p-1$, where $\alpha = \varepsilon_1 - \varepsilon_{i+1}$. This shows that the block in question contains at least all the p -hooks.

To finish, we need to show that it contains no other partition. Suppose that $\mu \vdash p$ is not a p -hook. Equivalently, $\mu_2 \geq 2$. This forces $\mu_1 \geq 2$ and $\mu_{p-1} = \mu_p = 0$ as well. If $\mu_2 = 2$ then μ cannot satisfy the criterion in the first paragraph since the modulo p residue of $\mu + \rho$ in place 2 is zero and matches the modulo p residue of $\mu + \rho$ in place p . If $\mu_2 = 3$ then the last three parts of μ must be zero and the modulo p residue of $\mu + \rho$ in place 2 is 1 and matches the modulo p residue of $\mu + \rho$ in place $p-1$. The argument continues in this way. \square

Theorem. $\Delta(\lambda^i)$ has two composition factors $L(\lambda^i)$, $L(\lambda^{i+1})$ for all $0 \leq i < \min(n-1, p-1)$. If $i = \min(n-1, p-1)$ then $\Delta(\lambda^i)$ is simple.

Proof. First we assume that $n \geq p$. We apply the Schur functor to $\Delta(\lambda^i)$. By (1) we have $e\Delta(\lambda^i) \simeq S_{\lambda^i}$. By a theorem of Peel [20] (see also [13, Theorem 24.1]), for $p > 2$ the dual Specht modules labeled by p -hooks λ^i have at most two composition factors. More precisely, S_{λ^i} has exactly two composition factors if $0 < i < p-1$ and just one if $i = 0$ or $p-1$.

By Lemma 2 and the isomorphism (3), it follows that for $0 < i < p-1$ no composition factor of $\Delta(\lambda^i)$ is killed by the Schur functor. Thus it follows from Peel's result that the $\Delta(\lambda^i)$ must have at most two composition factors when $p > 2$. Combining this with Lemma 1, the Theorem follows for all $0 < i \leq p-1$ in case $p > 2$.

In case $p = 2$ or $i = 0$ the result is easy to prove directly. For instance, one may apply the main result of [7] to $\nabla(\lambda^0)$ since this is isomorphic with a symmetric power of the natural module.

Now that the result has been established in case $n \geq p$, we consider the case $n < p$. In this case there is an idempotent Schur functor sending $S(p, p)$ -mod to $S(n, p)$ -mod. By the results in [11, §6.5] we obtain the result in general. \square

Remark. The Theorem also follows from the character formula of [2]. Take $0 \leq i \leq \min(n, p-1)$ as in the Theorem. The dimension of the μ weight space of $\Delta(\lambda^i)$ is equal to the number of standard λ^i -tableaux of weight μ . The latter set is the disjoint union of the set of standard λ^i -tableaux with no 'bad' p -hook (in the sense of §6 of [2]) and the image of the set of standard λ^{i+1} -tableaux of weight μ with no bad p -hook under the map sending a tableau of shape λ^{i+1} to one of shape

λ^i by moving its bottom node up to the end of its top row. This shows that $\text{ch } \Delta(\lambda^i) = \text{ch } L(\lambda^i) + \text{ch } L(\lambda^{i+1})$, as desired.

We now give some consequences of the Theorem.

Corollary 1. *We have $\text{ch } L(\lambda^i) = \sum_{j \geq i} (-1)^{j-i} \text{ch } \Delta(\lambda^j)$ for all $0 \leq i \leq \min(n-1, p-1)$.*

Proof. This follows immediately from the Theorem, by induction on i starting with the base case $i = \min(n-1, p-1)$ and working backwards in i . \square

Corollary 2. *For all $\mu \vdash p$ with at most n parts and all $0 \leq i \leq \min(n-1, p-1)$ we have:*

$$\text{Ext}_{\text{GL}_n(k)}^1(L(\lambda^i), L(\mu)) \simeq k$$

in case:

- (i) $0 < i < \min(n-1, p-1)$ and $\mu = \lambda^{i+1}$ or λ^{i-1} ;
- (ii) $i = 0$ and $\mu = \lambda^1$;
- (iii) $i = \min(n-1, p-1)$ and $\mu = \lambda^{i-1}$.

For all other cases $\text{Ext}_{\text{GL}_n(k)}^1(L(\lambda^i), L(\mu)) = 0$.

Proof. It is known that for partitions $\lambda \vdash r, \mu \vdash r$ we have isomorphisms (for any r)

$$\text{Ext}_{\text{GL}_n(k)}^1(L(\lambda), L(\mu)) \simeq \text{Ext}_{\mathbf{M}_n(k)}^1(L(\lambda), L(\mu)) \simeq \text{Ext}_{S(n,r)}^1(L(\lambda), L(\mu)).$$

The result now follows from the Theorem by Lemma 2 and [14, Part II, (2.14)(4)]. \square

Given a partition λ of length at most n we denote by $P(\lambda)$ (respectively, $I(\lambda)$) the projective hull (respectively, injective envelope) of $L(\lambda)$ in the category $\mathbf{M}_n(k)\text{-mod}$. We have $P(\lambda) \simeq {}^\tau I(\lambda)$. Recalling the Mullineux map [19], by [3, Lemma 3.3] we have that

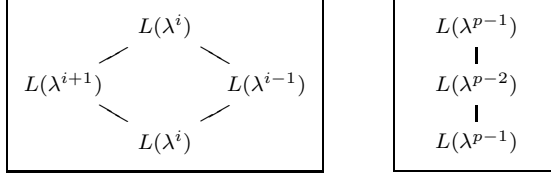
$$(4) \quad I(\lambda) = T(\text{Mull}(\lambda')) \text{ if } \lambda \text{ is column } p\text{-regular.}$$

Moreover, in that case $\text{Mull}(\lambda')$ is again a partition of length at most n . We recall from [4, (2.2h)] that $I(\lambda)$ has a ∇ -filtration and for all partitions λ, μ of length $\leq n$ the number $(I(\lambda) : \nabla(\mu))$ of subquotients in the filtration isomorphic with $\nabla(\mu)$ satisfies the reciprocity law

$$(5) \quad (I(\lambda) : \nabla(\mu)) = [\nabla(\mu) : L(\lambda)]$$

where the number on the right-hand side stands for the composition factor multiplicity of $L(\lambda)$ in a composition series of $\nabla(\mu)$. Another corollary of our main result is the following.

Corollary 3. *Suppose that $p > 2$. The module structure of $P(\lambda^i) = I(\lambda^i) = T(\lambda^{i-1})$ is as follows, for all $1 \leq i \leq \min(n-1, p-1)$ (the left diagram is for the case $i < \min(n-1, p-1)$ and the right one for $i = \min(n-1, p-1)$):*



where the module diagram is interpreted as described in [1]. For $p = 2$ the module structure of $P(\lambda^1) = I(\lambda^1) = T(\lambda^0)$ is depicted in the right diagram above.

Proof. From Lemma 2, the reciprocity law (5), and the Theorem it follows that for each i , $1 \leq i \leq p-1$, the module $I(\lambda^i)$ has a ∇ -filtration with subquotients $\nabla(\lambda^i)$ and $\nabla(\lambda^{i+1})$, each occurring with multiplicity one. From this and Corollary 2 it follows that the module structure of $I(\lambda^i)$ must be as described in all cases.

Since the module $I(\lambda^i)$ is contravariantly self-dual, i.e., $I(\lambda^i) \simeq {}^\tau I(\lambda^i)$, it follows immediately that $I(\lambda^i) = P(\lambda^i)$. Hence it follows that $I(\lambda^i) = P(\lambda^i)$ is a tilting module. Its highest weight is λ^{i-1} , so we conclude that $I(\lambda^i) = P(\lambda^i) = T(\lambda^{i-1})$. This completes the proof. \square

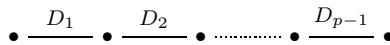
Remarks. 1. By the equality (4) it follows from the last corollary that $\text{Mull}((\lambda^i)') = \lambda^{i-1}$. This may also be checked combinatorially.

2. From (5) it also follows that $I(\lambda^0) \simeq \nabla(\lambda^0)$ and thus that $P(\lambda^0) \simeq \Delta(\lambda^0)$. These modules are not tilting.

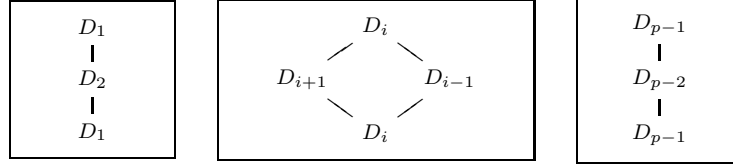
3. COMPARISON WITH SYMMETRIC GROUPS

It will be illuminating to compare our results with well known results concerning blocks of cyclic defect group for symmetric groups. We are concerned with the group Σ_p in characteristic p , which has just one block $\mathbf{B} = k\Sigma_p$.

For convenience we assume first that $p > 2$. We label the simple \mathbf{B} -modules by the column p -regular partitions λ^i for $1 \leq i \leq p-1$ and denote them by $D_i = D_{\lambda^i}$. (The reader who prefers to label by p -regular partitions should use the isomorphism $D_{\lambda^i} \simeq D^{\lambda^{i-1}}$.) As follows from [20], the Brauer tree of \mathbf{B} is an open polygon with no exceptional vertex, as depicted in the figure below.



The edges of the tree are in one-one correspondence with the simple \mathbf{B} -modules; the simple modules appearing in order as $D_i = D_{\lambda^i}$ for $1 \leq i \leq p-1$. (note $D_{p-1} = D_{(1^p)}$), see [16, §2.2]. Let P_i be the projective cover of D_i . Then the first diagram below gives the module structure of P_1 , the second is P_i (for all $2 \leq i \leq p-2$) and the third is P_{p-1} :



where again the diagrams are to be interpreted as in [1].

For $p = 2$ there is just one simple (namely, the trivial module) and its projective cover is uniserial of length 2.

Assume that $n \geq p$. Let us denote by \mathbf{H} the block of $\mathbf{M}_n(k)$ containing the simple modules labeled by the p -hooks λ^i for $0 \leq i \leq p-1$. Comparing the results in Corollary 3 with the description of the block \mathbf{B} in this section, we see that \mathbf{H} and \mathbf{B} are nearly equivalent, in the sense that \mathbf{H} contains one more simple (and thus one more projective) than does \mathbf{B} and upon deleting all references to the offending simple module (and its projective) from \mathbf{H} we recover \mathbf{B} . This deletion procedure is precisely the effect of the Schur functor applied to \mathbf{H} .

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